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NONLINEAR DEFORMATION AND STABILITY OF ELLIPTIC CYLINDRICAL SHELLS UNDER TORSION AND BENDING

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The problem of nonlinear deformation and buckling of noncircular cylindrical shells under combined loading is solved by the variational finite-element method in the displacement formulation. A numerical algorithm for solving the problem is proposed. Stability of cylindrical shells with an elliptic cross-sectional contour under a combined action of torsion and bending is analyzed. The effect of cross-sectional ellipticity and nonlinear prebuckling deformation on the critical loads and buckling mode is studied.

Key words: *elliptic cylindrical shells, combined torsion and bending, nonlinear deformation, stability, finite-element method.*

1. Finite Element and Algorithm for Solving the Problem. We consider a cantilevered ($u = v = w = \partial w/\partial x = 0$) noncircular cylindrical shell under the action of torque $M_{\rm t}$ and bending moment M applied to the end of the shell (Fig. 1). The loaded end of the shell is reinforced by a frame rigid in its plane. The torque is modeled by the tangential load $S = M_{\rm t}/(2\omega)$ acting per unit length ($\omega = \pi ab$ is the area bounded by the shell cross section and a and b are the semiaxes of the ellipse). The bending moment M is modeled by the axial load distributed nonuniformly over the shell circumference $T = Mz_1/J$ (z_1 is the distance from the points of the shell contour to the ellipse axis AA and J is the cross-sectional moment of inertia about the axis AA).

We divide the shell by the principal-curvature lines into m and n parts along the generatrix and directrix, respectively. Thus, the shell is modeled by $m \times n$ curvilinear rectangular finite elements. Using bilinear approximation for tangential deformation displacements and bicubic approximation for deflections and finite-element displacement as rigid bodies, we write the total displacements of finite-element points in the form

$$u = a_1xy + a_2x + a_3y + a_4 + a_6\psi_2 + a_{20}\psi_1,$$

$$v = a_5xy + a_6xc + a_7y + a_8(\psi_1c + \psi_2s) - a_{20}xs + a_{23}c - a_{24}s,$$

$$w = a_9x^3y^3 + a_{10}x^3y^2 + a_{11}x^3y + a_{12}x^3 + a_{13}x^2y^3 + a_{14}x^2y^2 + a_{15}x^2y + a_{16}x^2 \tag{1.1}$$

$$+ a_{17}xy^3 + a_{18}xy^2 + a_{19}xy + a_{20}xc + a_{21}y^3 + a_{22}y^2 + a_{23}s + a_{24}c + a_6xs + a_8(\psi_1s - \psi_2c).$$

For an arbitrary shell [1], we have $\psi_1 = \displaystyle\int Rs\,d\beta$, $\psi_2 = -\displaystyle\int Rc\,d\beta$, where $c = \cos\beta$, and $s = \sin\beta$, and R is the curvature radius of the shell contour.

For an elliptic shell, we obtain

$$\frac{z^2}{b^2} + \frac{y^2}{a^2} = 1, \quad R = \frac{a^2b^2}{d^3}, \quad d^2 = a^2s^2 + b^2c^2, \quad \psi_1 = -\frac{b^2c}{d}, \quad \psi_2 = -\frac{a^2s}{d}.$$

Relations (1.1) can be written in the matrix notation as

$$\tilde{\boldsymbol{u}} = P\boldsymbol{a}, \tag{1.2}$$

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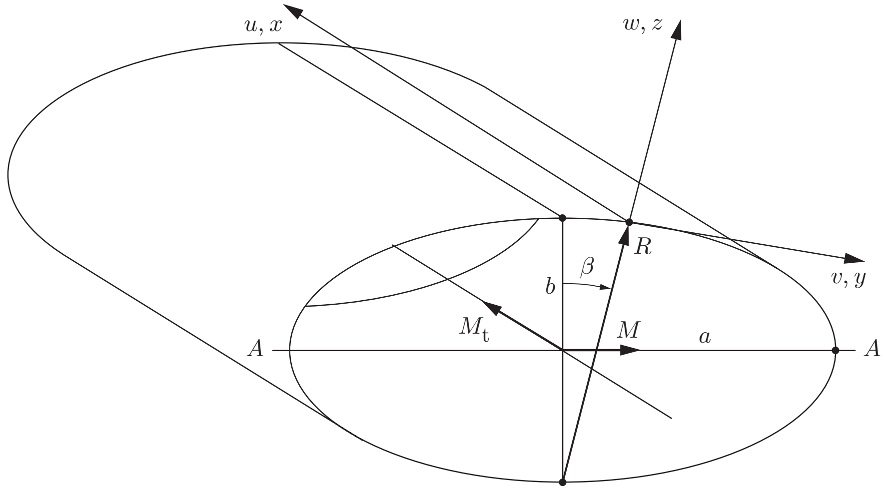

Fig. 1

where $\tilde{\boldsymbol{u}} = \{u, v, w\}^{\mathrm{t}}$ is the displacement vector of the points of the mid-surface of the finite element, $\boldsymbol{a} = \{a_1, \ldots, a_{24}\}^{\mathrm{t}}$ is the vector of unknown coefficients a_i, and P is the 3×24 coupling matrix whose elements are the multipliers p_{ij} at the coefficients a_i in functions (1.1). Expressing the coefficients a_i in terms of nodal unknowns, we obtain

$$\boldsymbol{a} = B^{-1}\bar{\boldsymbol{u}}. \tag{1.3}$$

Here $\bar{\boldsymbol{u}} = \{u_i, v_i, w_i, \vartheta_{1i}, \vartheta_{2i}, w_{xyi}, u_j, v_j, w_j, \vartheta_{1j}, \vartheta_{2j}, w_{xyj}, u_k, \ldots, w_{xyk}, u_n, \ldots, w_{xyn}\}^{\mathrm{t}}$ is the vector of nodal displacements, angles of rotation, and mixed derivatives of deflection, and B is a 24×24 matrix whose nonzero elements have the form

$$b_{1,j} = p_{1,j}, \qquad b_{2,j} = p_{2,j}, \qquad b_{3,j} = p_{3,j}, \qquad b_{4,j} = (p_{3,j})_x,$$

$$b_{5,j} = (p_{2,j} - (p_{3,j})_y)/R, \qquad b_{6,j} = (p_{3,j})_{xy} \qquad (x = -a_1, \quad y = -b_1),$$

$$b_{7,j} = p_{1,j}, \qquad b_{8,j} = p_{2,j}, \qquad b_{9,j} = p_{3,j}, \qquad b_{10,j} = (p_{3,j})_x,$$

$$b_{11,j} = (p_{2,j} - (p_{3,j})_y)/R, \qquad b_{12,j} = (p_{3,j})_{x\beta} \qquad (x = -a_1, \quad y = b_1),$$

$$b_{13,j} = p_{1,j}, \qquad b_{14,j} = p_{2,j}, \qquad b_{15,j} = p_{3,j}, \qquad b_{16,j} = (p_{3,j})_x,$$

$$b_{17,j} = (p_{2,j} - (p_{3,j})_\beta)/R, \qquad b_{18,j} = (p_{3,j})_{xy} \qquad (x = a_1, \quad y = -b_1),$$

$$b_{19,j} = p_{1,j}, \qquad b_{20,j} = p_{2,j}, \qquad b_{21,j} = p_{3,j}, \qquad b_{22,j} = (p_{3,j})_x,$$

$$b_{23,j} = (p_{2,j} - (p_{3,j})_y)/R, \qquad b_{24,j} = (p_{3,j})_{xy} \qquad (x = a_1, \quad y = b_1),$$

$$j = 1, \ldots, 24, \qquad a_1 = L/(2m), \quad b_1 = l/(2n)$$

(L and l are the characteristic lengths of the shell along the generatrix and directrix, respectively; the subscripts x, y, and β denote partial differentiation with respect to x, y, and β, respectively).

Substituting (1.3) into (1.2), we obtain the relation between the displacements of element points and nodal unknowns:

$$\tilde{\boldsymbol{u}} = PB^{-1}\bar{\boldsymbol{u}}.$$

There are six unknowns at each node and, hence, the finite element has 24 degrees of freedom. To determine the nodal unknowns, we use the Lagrange variational equation $\delta\Pi = 0$, where Π is the potential energy of the shell. The potential energy is written in terms of nonlinear strain–displacement relations [2]. The equation $\delta\Pi = 0$ leads to a system of nonlinear algebraic equations for nodal unknowns. This system is solved by the step-by-step method by varying the load. At each step, we use the Newton–Kantorovich linearization method; its equation can be written in the form [3]

$$H(\bar{\boldsymbol{u}}^n)\Delta\bar{\boldsymbol{u}} = \boldsymbol{q}_e - \boldsymbol{G}(\bar{\boldsymbol{u}}^n), \qquad \bar{\boldsymbol{u}}^{n+1} = \bar{\boldsymbol{u}}^n + \Delta\bar{\boldsymbol{u}}, \tag{1.4}$$

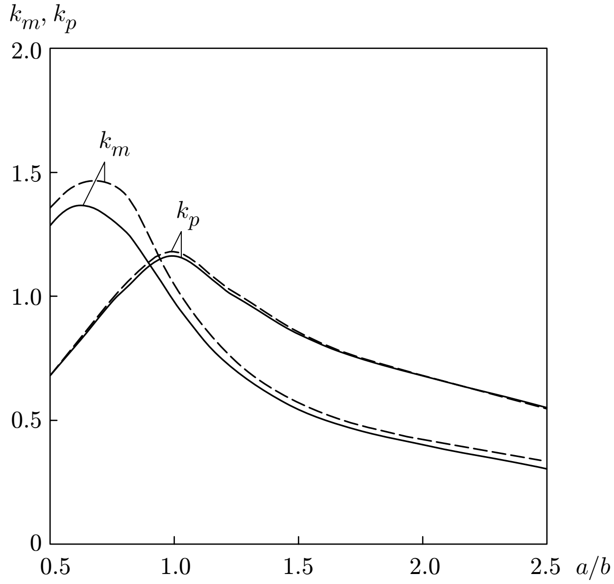

Fig. 2

where H is the Hessian matrix of the shell determined by the second variation of the potential strain energy, $\boldsymbol{q}_e$ is the vector of the nodal load, and $\boldsymbol{G}$ is the gradient of the potential strain energy. Equations (1.4) are constructed in a standard manner with allowance for boundary conditions [4]. The boundary conditions are formulated as follows. For zero nodal boundary displacements, the corresponding row of the Hessian matrix H of the boundary element and the corresponding element of the nodal-load vector are set equal to zero, and the diagonal coefficient in the matrix H is replaced by a large number.

The system of linear algebraic equations (1.4) is solved by the Kraut method using decomposition of the Hessian matrix $H = L^{\mathrm{t}} DL$ (D is the diagonal matrix and L is the triangular matrix). Once the nodal displacements are determined, the stresses and strains are calculated by the known formulas of [2]. Stability is controlled by investigating the positive definiteness of the Hessian matrix, which reduces to verifying that the elements of the diagonal matrix D are positive. The appearance of negative elements means that the shell is unstable. After the loading parameter for which the equilibrium state becomes unstable is calculated, we determine the buckling mode of the shell by solving the system $H\boldsymbol{\delta} = 0$, where $\boldsymbol{\delta}$ is the vector of bifurcational nodal displacements. To this end, we find the row of the matrix H that corresponds to the first negative element of the matrix D. This row and the corresponding column of the matrix H are deleted. We replace the diagonal coefficient by unity, multiply the corresponding column by the subcritical displacement that refers to the degenerated row and move this column to the right side of the system. Solving the resulting system, we obtain the buckling mode of the shell.

2. Results of Numerical Analysis of Nonlinear Deformation and Stability of Elliptic Shells. Calculations were performed for the following parameters: shell length $L = 500$ and 1100 mm, thickness $h = 5$ mm, Young's modulus $E = 0.7 \cdot 10^5$ MPa, Poisson's ratio $\nu = 0.3$, and equiperimeter radius (cross-sectional radius of a circular shell with a perimeter P equal to the perimeter of the elliptic shell) $R_0 = 1000$ mm. The value of R_0 was calculated by the formula

$$R_0 = \frac{P}{2\pi} = \frac{2a}{\pi} \int\limits_0^{\pi/2} \Big\{1 + \Big[\Big(\frac{b}{a}\Big)^2 - 1\Big] \sin^2\psi\Big\}^{1/2} \delta\psi = \frac{2a}{\pi} E\Big(\frac{\pi}{2}, \frac{b}{a}\Big),$$

where $E(\pi/2, b/a)$ is the complete elliptic integral of the second kind.

Figure 2 shows the parameters $k_m = M^*/M_0$ and $k_p = M_{\mathrm{t}}^*/M_{\mathrm{t}0}$ versus the ellipticity parameter a/b for the linear and nonlinear prebuckling states (dashed and solid curves, respectively) of a shell with $h = 5$ mm and $L = 500$ mm produced by a separate action of the moments [M^* and M_{t}^* are the critical bending moment and torque, respectively, and $M_0 = \pi E R_0 h^2/\sqrt{3(1-\nu^2)}$ and $M_{\mathrm{t}0} = 2\pi C R_0^2 S_b$ are the critical bending moment and torque for the equiperimeter circular cylindrical shell, respectively, $S_b = 0.78Eh(h/R_0)^{5/4}(R_0/L)^{1/2}$, and $C = 0.953$]. As the

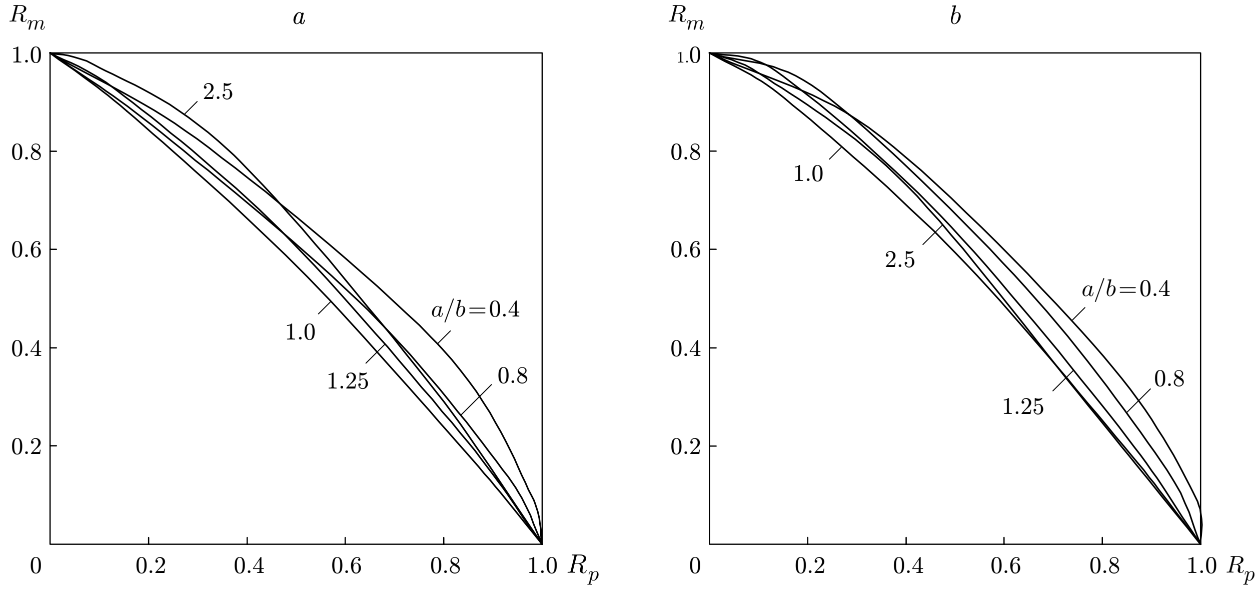

Fig. 3

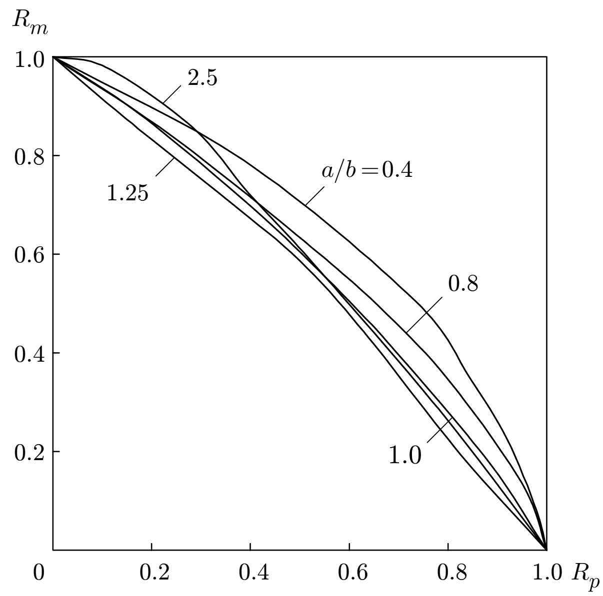

Fig. 4

ellipticity increases, the values of k_p decrease almost proportionally to the ratio of the minor semiaxis to the major semiaxis. The most stable shell is the equiperimeter circular shell ($k_p = 1.16$). The same effect is observed for bending of flattened shells ($a/b > 1$). High shells ($a/b < 1$) were found to be more stable compared to equiperimeter circular shells due to the higher cross-sectional moment of inertia (double–tee effect). The most stable shells are those with $a/b \approx 0.7$ ($k_m = 1.34$). Nonlinearity affects only slightly. For bending, the effect of nonlinearity becomes more pronounced: the difference between the critical moments reaches 10%.

Figure 3a and b shows the curves $R_m(R_p)$ obtained, respectively, for linear and nonlinear prebuckling stress–strain states of the shells with $h = 5$ mm and $L = 500$ mm for various values of the ellipticity parameter ($R_m = k_m/k_{m0} = M^*/M_p^*$ and $R_p = k_p/k_{p0} = M_{\rm t}^*/M_{{\rm t},p}^*$, where k_{m0}, k_{p0} and M_p^* and $M_{{\rm t},p}^*$ are the critical values of the parameters k_m and k_p and moments M and $M_{\rm t}$ for separate loading). One can see from Fig. 3 that the nonlinearity of the prebuckling state affects the dependence $R_m(R_p)$ only slightly.

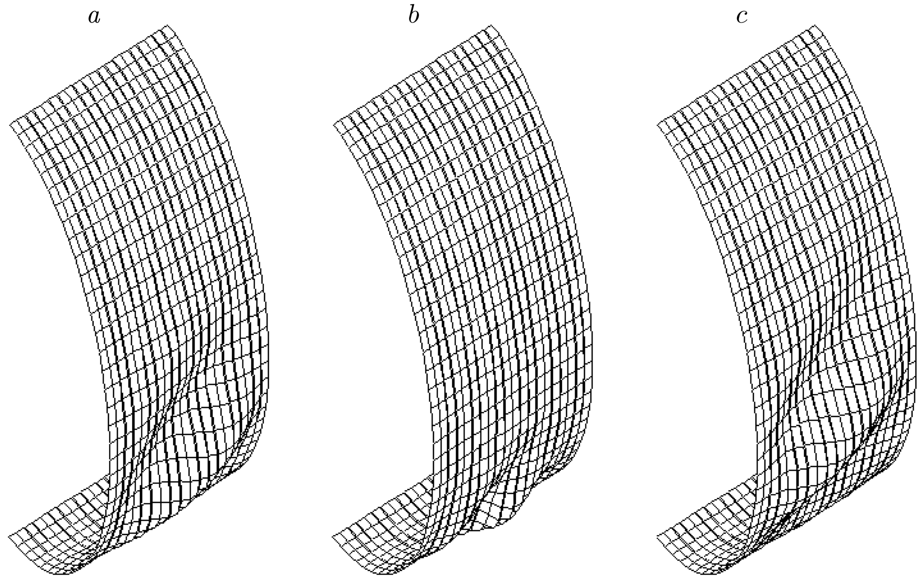

Fig. 5

Figure 4 shows the curves $R_m(R_p)$ for a long shell ($h = 5$ mm and $L = 1100$ mm) that refer to the linear prebuckling stress–strain state.

The buckling mode of shells depends strongly on the length, ellipticity parameter, and ratio k_m/k_p. For $k_m/k_p < 1$ and $a/b < 1$, one to three oblique wrinkles are formed on the lateral surface of the shell upon buckling. As the ratios a/b and k_m/k_p increase, wrinkling is shifted from the lateral surface to the lower part of the shell, and the number of waves decreases from three to one. For $k_m/k_p < 1$, high shells buckle under the action of the tangential forces, and three oblique wrinkles are formed. For $k_m/k_p > 1$, buckling occurs in the lower part of the shells due to the maximum compressive axial stresses and diamond-shaped dents are formed. Figure 5 shows the buckling modes of the shell with $L = 1100$ mm, $h = 5$ mm, and $a/b = 0.4$ under torsion (a), bending (b), and combined action of bending and torsion ($k_m/k_p = 1$) (c).

The results described above were obtained using a finite-element mesh that ensured convergence of the solution.

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